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OPTIMAL DAMPING AND STOCHASTIC CONTROL IN CERTAIN ASTRODYNAMIC PROBLEMS

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16. Abstract

On the basis of controllable Markovian random processes, a theory of optimal damping is constructed. Interference, in this context, pertains to white noise. The application of this theory to the problem of retarding rotation of an aircraft is given.

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OPTIMAL DAMPING AND STOCHASTIC CONTROL IN CERTAIN AERODYNAMIC PROBLEMS

V. T. Tarushkin

On the basis of methods of controllable Markovian random processes is constructed a theory of optimal damping. The case where interference is a process of the white noise type is discussed individually and for it is given a derivation of all main relationships of the theory of random processes. The application of this theory to the problem of retarding rotation of an aircraft is given. /114

1. Methods of stochastic control and the theory of optimal damping

Given that an n-dimensional vectorial Markovian diffusion process y satisfies a system of stochastic differential equations in the Ito form:

$$\boxed{dy(t)=f[t,\ y(t),\ u]\,dt+\sigma[t,\ y(t),\ u]\,dw.} \tag{1.1}$$

Here f is the n-dimensional vector of transposition, an σ-matrix of diffusion of dimensionality $n \times m$, where w is an m-dimensional process of Brownian motion with a unique dispersion parameter, $u[t, y(t)]$ is an r-dimensional vector of control effects from a given set U of permissible piecewise-continuous functions. Limitations on transposition, diffusion and control are given such that equation (1.1) for several intervals of $[\tau_0, \tau] \subset [0, T]$ is equivalent to the integral equation

$$\boxed{\begin{aligned} y(\tau)=y(\tau_0)+\int_{\tau_0}^{\tau} f[s,\ y(s),\ u]\,ds+ \\ +\int_{\tau_0}^{\tau} \sigma[s,\ y(s),\ u]\,dw(s). \end{aligned}} \tag{1.2}$$

For equation (1.2) are assumed fulfilled the conditions of the theorem of existence and uniqueness of the diffusion Markovian process in the interval $[\tau_0, \tau]$ with initial conditions $y(\tau_0)$ [1-2]. It is also assumed that the interval $[0, T]$ can be discovered with the aid of a finite number of intervals of the form $[\tau_0, \tau]$, where τ_0 is in identity with any moment of motion along the trajectory, and --with the moment of acquisition of the surface of reversing by the system. Under the condition that for each moment $t \in [0, T]$ it known an observable value of y_t of the process $y(t)$, we must find the minimal value of the functional /115

$$E\left\{\int_0^T L[\tau, y(\tau), u[\tau, y(\tau)]]\, d\tau / y(0) = y_0\right\}. \tag{1.3}$$

for $u \in U$. Here E is a symbol of mathematical expectation.

Let us introduce the function

$$V[t, y(t)] = \int_t^T L[\tau, y(\tau), u[\tau, y(\tau)]]\, d\tau,$$

and also define the σ-algebra of $Y_t = \sigma\{y_s, 0 \leq s \leq t\}$, generated by the values of y_s for $0 \leq s \leq t$. Due to the fact that for any $t \in [0, T]$ $y_t = y(t)$,

$$E\{V[t, y(t)]/Y_t\} = E\{V[t, y_t]/Y_t\}.$$

Since y_t is measured with respect to Y_t, theorem 8.3 [3] (p. 27) yields the condition at which with a probability of 1

$$E\{V(t, y_t)/Y_t\} = V(t, y_t).$$

The condition of applicability of this theorem consists of the fact that function $V(t, y_t)$ is for $t \in [0, T]$ measurable with respect to Y_t, and also $E|V(t, y_t)| < \infty$.

Therefore, with a probability of 1 takes place

$$V(t, y_t) = E\left\{\int_t^T L[\tau, y(\tau), u[\tau, y(\tau)]]\,d\tau/Y_t\right\}. \tag{1.4}$$

Relationship (1.4) is a stochastic function of Lyapunov [4-5] for which

$$V(T, y_T) = 0. \tag{1.5}$$

Let us find the optimal control based on the Bellman equation:

$$\min_{u \in U}\left[\frac{\partial V}{\partial t} + f'\frac{\partial V}{\partial y_t} + \frac{1}{2}\,tr\,\frac{\partial^2 V}{\partial y_t^2}\sigma\sigma' + L\right] = 0. \tag{1.6}$$

Here the prime ' signifies transposition, and the symbol tr--the matrix spur.

From equation (1.6) it follows that if the diffusion matrix σ and function L do not depend on control, the minimum in (1.6) is attained when the minimum of function

$$F = f'\frac{\partial V}{\partial y_t}. \tag{1.7}$$

is also attained. The relation (1.7) is introduced in [6] as a criterion of optimal damping of a determinate Lyapunov function. To substantiate relations (1.6)-(1.7), let us apply the method systematically developed in [4-5] and prove the theorems which /116 define optimal control.

Let us introduce the operator A(u) which applies to the function V yielding

$$A(u)V_f = \frac{\partial V}{\partial t} + \frac{1}{2}\,tr\,\frac{\partial^2 V}{\partial y_t^2}\sigma\sigma' + f'\frac{\partial V}{\partial y_t}. \tag{1.8}$$

With the aid of (1.8), the Ito differential formula [2] is written in the form

$$dV(t, y_t) = \{A(u)V\}\,dt + V_{y_t}\sigma\,dw. \tag{1.9}$$

Integrating (1.9) and taking the arbitrary mathematical expectation, we arrive at the integral formula of Dynkin-Ito [4-5], which with allowance for (1.5) is written as

$$E\{-V(t, y_t)/Y_t\} = E\left\{\int_t^T A(u)\,V\,d\tau/Y_t\right\}. \tag{1.10}$$

Given $u_0 \in U$ satisfies equation (1.6), then takes place for $t \in [0, T]$

$$A(u_0)V + L[t, y_t, u_0] \leqslant A(u)V + L[t, y_t, u]. \tag{1.11}$$

Integrating (1.11) and taking the arbitrary mathematical expectation with allowance for (1.10), we find that

$$E\left\{\int_t^T L[\tau, y, u_0]\,d\tau/Y_t\right\} \leqslant E\left\{\int_t^T L[\tau, y, u]\,d\tau/Y_t\right\}, \tag{1.12}$$

or due to (1.4) we have

$$V[t, y_t(u_0)] \leqslant V[t, y_t(u)],\ t \in [0, T]. \tag{1.13}$$

Therefore, the following assertion is proven.

Theorem: The condition of optimal damping is a sufficient criterion for finding the optimal control under the condition that the integrand of the function and the diffusion matrix are not explicitly control-dependent.

2. Control in the Presence of White Noise

Let us introduce the Gaussian random process $v[t, y(t)]$ of the white noise type, having a zero mean of

$$E\{v(t, y)/y(t)=y_t\}=0 \tag{2.1}$$

and a covariation matrix

$$E\{v[t, y(t)]v[s, y(s)]' \mid y(t)=y_t\}=R(t, y_t)\delta(t-s), \tag{2.2}$$

where $\delta(t - s)$ is a Dirac function, $R = \sigma\sigma'$. We will show that if (1.2) underoges a formal substitution of

$$\sigma\frac{dw}{dt}=v, \tag{2.3}$$

the basic relations of the preceeding section can be derived on the basis of ordinary analytic methods of generalized Dirac functions. /117

Equation (1.2) is written as follows for this case

$$y(\tau)=y(\tau_0)+\int_{\tau_0}^{\tau} f\,ds+\int_{\tau_0}^{\tau} v\,ds. \tag{2.4}$$

Assuming in (2.4) that $\tau_0 = t, \tau = t + \Delta t$, $\Delta y_t = y(t + \Delta t) - y(t)$, and taking into account (2.1) and (2.2), we find that

$$\begin{gathered}\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\{\Delta y_t/y(t)=y_t\}=f[t, y_t, u(t, y_t)],\\ \lim_{\Delta t\to 0}\frac{1}{\Delta t}E\{\Delta y_t\Delta y_t'/y(t)=y_t\}=R(t, y_t).\end{gathered} \tag{2.5}$$

Let us introduce the funtion of risk

$$S(t, y_t)=E\left\{\int_t^T L[\tau, y(\tau), u[\tau, y(\tau)]]\,d\tau/y(t)=y_t\right\}. \tag{2.6}$$

Function (2.6) has the same meaning as (1.4). It is easy to see that with a probability of 1 takes place

$$
\begin{gathered}
E\{S(t, y_t)/y(t)=y_t\}=S(t, y_t), \quad E\left\{\frac{\partial^2 S}{\partial y_t \partial t} \Big/ y(t)=y_t\right\}=\frac{\partial^2 S}{\partial y_t \partial t}, \\
E\left\{\frac{\partial S(t, y_t)}{\partial t} \Big/ y(t)=y_t\right\}=\frac{\partial S(t, y_t)}{\partial t}, \\
E\left\{\frac{\partial S(t, y_t)}{\partial y_t} \Big/ y(t)=y_t\right\}=\frac{\partial S(t, y_t)}{\partial y_t}, \\
E\left\{\frac{\partial^2 S(t, y_t)}{\partial y_t^2} \Big/ y(t)=y_t\right\}=\frac{\partial^2 S(t, y_t)}{\partial y_t^2}.
\end{gathered} \tag{2.7}
$$

These properties follow from the definition of the risk function and can be formally proven with the aid of the aforementioned theorem 8.3 [3].

Let us consider the expansion of the risk function into a Taylor series up to terms of the second power exclusively

$$
\begin{gathered}
S(t+\Delta t, y_t+\Delta y_t)=S(t, y_t)+\frac{\partial S(t, y_t)}{\partial t}\Delta t+ \\
+\Delta y_t' \frac{\partial S(t, y_t)}{\partial y_t}+\frac{1}{2}\Delta y_t' \frac{\partial^2 S(t, y_t)}{\partial y_t \partial t}\Delta t+ \\
+\frac{1}{2}\frac{\partial^2 S(t, y_t)}{\partial t^2}\Delta t^2+\frac{1}{2} tr \frac{\partial^2 S(t, y_t)}{\partial y_t^2}\Delta y_t \Delta y_t'.
\end{gathered} \tag{2.8}
$$

It is easy to see that relationship (2.8), although it externally resembles (1.9), does not correspond to it, since white noise as a generalized derivative of Brownian motion is defined by equation /118
(2.2) and not by the formal relationship

$$\Delta y_t = f\Delta t + v\Delta t$$

From (2.8), with allowance for (2.5), (2.7), we find that

$$
\begin{gathered}
\lim_{\Delta t \to 0} E\left\{\frac{1}{\Delta t}[S(t+\Delta t, y_t+\Delta y_t)-S(t, y_t)/y(t)=y_t]\right\}= \\
=\frac{\partial S}{\partial t}+f'\frac{\partial S}{\partial y_t}+\frac{1}{2} tr \frac{\partial^2 S(t, y_t)}{\partial y_t^2} R.
\end{gathered} \tag{2.9}
$$

Let us note that in virtue of (1.8), the right side of (2.9) can be written in the form A(u)S. It is easy to see that since in (2.9) the arbitrary mathematical expectation is already taken, there takes place

$$E\{A(u)S/y(t)=y_t\}=A(u)S. \tag{2.10}$$

In virtue of (2.10), relationship (2.9) is rewritten as

$$E\left\{\frac{dS}{dt}\Big/y(t)=y_t\right\}=E\{A(u)S/y(t)=y_t\}. \tag{2.11}$$

Integrating (2.11) with allowance of the fact that $S(T, x_T) = 0$, we find that

$$E\{-S(t, y_t)/y(t)=y_t\}=E\left\{\int^T A(u)S\,d\tau/y(t)=y_t\right\}. \tag{2.12}$$

Relation (2.12) is analogous to the Dynkin-Ito formula (1.10).

In virtue of the principle of optimacy of Bellman, we find that

$$\begin{aligned} S(t, y_t)=\min_{u\in U} E\{L[t, y_t, u(t, y_t)]\Delta t+ \\ +S(t+\Delta t, y_t+\Delta y_t)/y(t)=y_t\}. \end{aligned} \tag{2.13}$$

Equation (2.13) in virtue of (2.7) is rewritten as

$$\begin{aligned} \min_{u\in U} E\{L[t, y_t, u(t, y_t)]\Delta t+S(t+\Delta t, y_t+\Delta y_t)- \\ -S(t, y_t)/y(t)=y_t\}=0. \end{aligned} \tag{2.14}$$

Let us divide both side of equation (2.14) by Δt and convert to the limit where $\Delta t \to 0$. In virtue of (2.9), we find the equation

$$\min_{u\in U}\left[\frac{\partial S}{\partial t}+f'\frac{\partial S}{\partial y_t}+\frac{1}{2}\,tr\,\frac{\partial^2 S}{\partial y_t^2}R+L\right]=0. \tag{2.15}$$

Equation (2.15) corresponds to (1.6).

If $u = u_0$ satisfies (2.15), then there takes place

$$A(u_0)S + L[t, y_t, u_0] \leqslant A(u)S + L[t, y_t, u]. \tag{2.16}$$

Integrating (2.16) and taking the arbitrary mathematical expectation, we derive with allowance of (2.12) that

$$E\left\{\int_t^T L(\tau, y, u_0)\,d\tau / y(t) = y_t\right\} \leqslant \left\{\int_t^T L(\tau, y, u)\,d\tau / y(t) = y_t\right\}. \tag{2.17}$$

Relationship (2.17) is analogous to (1.12).

Aside from the discussed analogy between the processes of white noise and Brownian movement, as a result of this section we can assert that if (2.13) is a necessary condition of optimacy, and the boundary value problem (1.5)-(1.6) has a unique solution, then the necessary and sufficient criteria of optimacy coincide. /119

3. Retardation of Aircraft Rotation

Let us denote through l_x, l_y, l_z the projections of kinetic momentum on an axis of a system of coordinates associated with an apparatus, and by u_x, u_y, u_z--projections of direction momentum on the same axis. Assuming that we have selected the main axes of coordinates, the equations of motion of the apparatus will be derived in the form

$$\begin{aligned} dl_x + \frac{(I_z - I_y)}{I_z I_y} l_y l_z\,dt &= u_x\,dt + \sigma_x\,dv_x, \\ dl_y + \frac{(I_x - I_z)}{I_x I_z} l_z l_x\,dt &= u_y\,dt + \sigma_y\,dv_y, \\ dl_z + \frac{(I_y - I_x)}{I_y I_x} l_y l_x\,dt &= u_z\,dt + \sigma_z\,dv_z. \end{aligned} \tag{3.1}$$

Here I_x, I_y, I_z are moments of inertia, $\sigma_x dv_x$, $\sigma_y dv_y$, $\sigma_z dv_z$ are projections of perturbing moment which are the products of constant coefficients of diffusion σ_x, σ_y, σ_z multiplied by increases in independent Brownian movements v_x, v_y, v_z.

In the regularly employed meters [7], projections of the angular velocities are measured. Considering that the measuring circuit contains elements which multiply the measurable signal by moments of inertia, we will consider that projections of kinetic momentum are being measured. Therefore, equations for the measurable coordinates have the form

$$x=l_x, \quad y=l_y, \quad z=l_z. \tag{3.2}$$

In the capacity of a function for optimal damping, we will select $V = \frac{1}{2}(x^2 + y^2 + z^2)$. The minimized relationship (1.7) in this case will be written as

$$J=xu_x+yu_y+zu_z. \tag{3.3}$$

Let us consider the following cases of limitations on the directional momentum.

1. Directional momentum is not limited. The scalar product of (3.3) will be minimal when the control vector is antiparallel to the vector $\{x, y, z\}'$, i.e., $u = \{-kx, -ky, -kz\}'$, $k = \mathrm{const} > 0$.

2. Each of the components of the directional momentum is limited in amplitude, i.e., $|u_x| \leq T_x$, $|u_y| \leq T_y$, $|u_z| \leq T_z$, where T_x, T_y, T_z are several constants. In this case /120

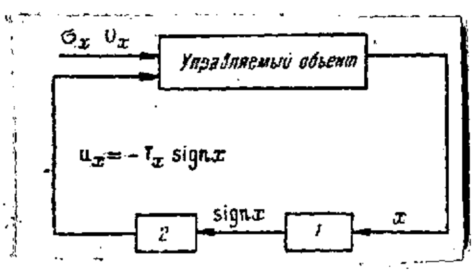
$$u_x=-T_x \operatorname{sign} x, \quad u_y=-T_y \operatorname{sign} y, \quad u_z=-T_z \operatorname{sign} z.$$

3. Directional momentum is limited in modulus, i.e., $|u| \leq T$, T = const. In this case

$$u_x = -T\frac{x}{\sqrt{x^2+y^2+z^2}}, \quad u_y = -T\frac{y}{\sqrt{x^2+y^2+z^2}},$$
$$u_z = -T\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

For the case where there is no interference in the measurement channel, the structure of the first two laws without research of optimacy is cited in [7]. The derivation of the third law in the absence of interference as control of rotation in terms of speed of response is cited in [8].

Let us note that the simplest structure of control is derived in the first instance. But it has a limited application, since in practice the directional moments are usually limited. The most complex is the structure of control derived in the third case, which requires for its realization the use of nonlinear signal transformers. The second case is most optimal from the standpoint of simplicity of realization allowing for the limitations. In this context, control of all three channels is done independently, while the structural circuits of the computers implementing the laws of control are identical. The structural diagram of a computer for this case, which realizes the law of control with respect to channel 1, is given in the figure.



Structural diagram of computer device implementing law of control with respect to channel 1.

The scheme of formation of the law of control includes a threshold element (1) at whose input enters signal x; at the output we have signal sign x, and also element (2) which multiplies the signal sign x by the number T_x.

Let us condider the case of absent interference, when $V = \frac{1}{2}(l_x^2 + l_y^2 + l_z^2)$, and $\sigma_x = \sigma_y = \sigma_z = 0$. It is easy to see that in virtue of (3.1)

$$\begin{aligned}\frac{dV}{dt} = l_x \dot{l}_x + l_y \dot{l}_y + l_z \dot{l}_z = \Big[\frac{I_y - I_z}{I_y I_z} l_x l_y l_z + \frac{I_z - I_x}{I_z I_x} l_x l_y l_z + \\ + \frac{I_x - I_y}{I_x I_y} l_x l_y l_z\Big] + l_x u_x + l_y u_y + l_z u_z. \end{aligned} \tag{3.4}$$

The expression in square brackets in (3.4) is transformed to the form /121

$$l_x l_y l_z \left[\frac{I_x I_y - I_x I_z + I_y I_z - I_y I_x + I_z I_x - I_z I_y}{I_x I_y I_z}\right]$$

and vanishes. Hence (3.4) is written as

$$\frac{dV}{dt} = l_x u_x + l_y u_y + l_z u_z. \tag{3.5}$$

Let us note that (3.5) has the same meaning as (3.3); for the three laws of control it is negative, ensuring decrease of kinetic momentum.

In the stochastic case, by formula (1.9) we find that

$$\begin{aligned}dV = \Big[\frac{I_y - I_z}{I_y I_z} xyz + \frac{I_z - I_x}{I_z I_x} xyz + \frac{I_x - I_y}{I_x I_y} xyz\Big] dt + \\ + [xu_x + yu_y + zu_z] dt + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) dt + \\ + x\sigma_x dv_x + y\sigma_y dv_y + z\sigma_z dv_z. \end{aligned} \tag{3.6}$$

Let us note that in (3.6) the expression in the first square brackets vanishes as before. Taking the arbitrary mathematical expectation in (3.6), we find that

$$E[dV/Y_t]=[xu_x+yu_y+zu_z]\,dt+\frac{1}{2}(\sigma_x^2+\sigma_y^2+\sigma_z^2) \qquad (3.7)$$

Relationship (3.7) shows that the rate of decrease in kinetic momentum is largely dependent on interference.

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